1D compaction

Part III Magma Dynamics: Practical 2

The compaction length

In one dimension, in the absence of body forces, McKenzie's two-phase flow equations can be simplified to

$$\frac{\mathrm{d}}{\mathrm{d}z}(q + v_s) = 0, \tag{1}$$

$$q = -\frac{k}{\mu}\frac{\mathrm{d}P}{\mathrm{d}z}, \tag{2}$$

$$-\frac{\mathrm{d}P}{\mathrm{d}z} + \left(\zeta + \tfrac{4}{3}\eta\right)\frac{\mathrm{d}^2 v_s}{\mathrm{d}z^2} = 0, \tag{3}$$

where $q \equiv \phi(v_f - v_s)$ is the Darcy flux, ϕ is the porosity, v_f is the melt velocity, v_s is the solid velocity, P is the melt pressure, k is the permeability, μ is the melt viscosity, ζ and η are the effective bulk and shear viscosities of the two-phase aggregate. (1) is a statement of conservation of mass, (2) is Darcy's law (conservation of momentum for the melt), and (3) is a statement of total conservation of momentum.

1. Show that (1), (2) and (3) can be combined to give a single equation for the compaction rate $\mathcal{C} = \mathrm{d}v_s/\mathrm{d}z$ as

$$-\frac{k}{\mu}\left(\zeta + \tfrac{4}{3}\eta\right)\frac{\mathrm{d}^2\mathcal{C}}{\mathrm{d}z^2} + \mathcal{C} = 0. \tag{4}$$

   What are the units of compaction rate?

2. Show that (4) defines a natural length scale (the compaction length) given by

$$\delta = \sqrt{\frac{k\left(\zeta + \tfrac{4}{3}\eta\right)}{\mu}}. \tag{5}$$

3. Suppose partially molten rock occupies the half-space $z \geq 0$, and a force is applied to the matrix at $z = 0$ such that $v_s = V_0$ at $z = 0$. Suppose that $v_s \to 0$ as $z \to \infty$. Integrate (4) to show that

$$v_s = V_0 \mathrm{e}^{-z/\delta}, \tag{6}$$

$$\mathcal{C} = -\frac{V_0}{\delta}\mathrm{e}^{-z/\delta}. \tag{7}$$

   (This problem is known as the coffee-press problem, and demonstrates that compaction takes place over a length scale δ near a boundary.)

The permeability of a partially molten rock varies with porosity. A simple model of this (based on melt in tubes) has

$$k = \frac{d^2\phi^2}{\mathrm{C}} \tag{8}$$

where d is the grain size and $\mathrm{C} = 1600$ is a geometric factor.

4. Estimate a typical compaction length during mantle melting, assuming $\zeta \approx \eta = 10^{19}$ Pa s, $d = 10^{-3}$ m, $\mu = 10$ Pa s, $\phi = 0.01$. How accurate do you think this estimate is?

5. Estimate a typical compaction length for a laboratory experiment, assuming $\zeta \approx \eta = 6 \times 10^{11}$ Pa s, $d = 10^{-5}$ m, $\mu = 10$ Pa s, $\phi = 0.02$. Lab equipment can apply shear stresses of around 50 MPa. To what strains can samples be deformed in a day? How do the strain rates in lab experiments compare with typical strain rates associated with mantle convection?

A 1D melting column

A simple model of mantle melting involves the uniform ascent of a column of rock, where melting occurs due to pressure release at constant entropy. The mass conservation equation (1) can be integrated as

$$q + v_s = V_0 \tag{9}$$

where V_0 is the upwelling velocity of material which enters at the base of the column (assumed pure solid). The total degree of melting is the ratio of the melt flux to the upwelling velocity,

$$F = \frac{\phi v_f}{V_0} \approx \frac{q}{V_0} \tag{10}$$

where a small-porosity approximation has been made. Assuming the pressure in the melt is approximately lithostatic, and neglecting compaction stresses, Darcy's law becomes

$$q = \frac{k}{\mu} \Delta \rho g \tag{11}$$

where $\Delta\rho = \rho_s - \rho_f$ is the density difference between solid and melt, and g is the gravitational acceleration.

6. If the mantle melts to a degree $F = 0.15$, what is the porosity ϕ? Assume $\mu = 10$ Pa s, $V_0 = 30$ mm yr^{-1}, $\Delta\rho = 500$ kg m^{-3}, $g = 10$ m s^{-2}, $d = 1$ mm. Assuming uniform melt productivity with depth, sketch a profile of porosity as a function of depth for a 1D melting column.

References

This compaction length concept was first introduced in:

McKenzie D. (1984) The generation and compaction of partially molten rock. *J. Petrol* **25** 713-765 doi:10.1093/petrology/25.3.713,

although the following article provides a more accessible (and shorter!) account:

McKenzie D. (1987) The compaction of igneous and sedimentary rocks. *J. Geol. Soc. Lond.* **144** 299-307 doi:10.1144/gsjgs.144.2.0299

The 1D melting column problem is described in:

Ribe N. M. (1985) The generation and composition of partial melts in the Earth's mantle. *Earth Planet. Sci. Lett.* **73** 361-376 10.1016/0012-821X(85)90084-6

Answers

1. Units of compaction rate are s^{-1}.

4. $\delta = 400$ m. There are reasonably large uncertainties in this estimate because the grain size is not well known, nor is the bulk viscosity (which might be an order of magnitude or two larger than the shear viscosity).

5. $\delta = 2$ mm. Laboratory strain rate $\dot{\epsilon}_{\rm lab} = \sigma/\eta \sim 10^{-4}\ \rm s^{-1}$. In a day this leads to a strain of 7. The strain rate in the convecting mantle is much much smaller, $\dot{\epsilon}_{\rm mantle} \sim 10^{-15}\ \rm s^{-1}$.

6. $\phi = \left(\dfrac{\mu F V_0 C}{\Delta\rho g d^2}\right)^{1/2} \sim 0.02$. If melt productivity is uniform with depth $F \propto z$, and hence $\phi \propto z^{1/2}$, so the sketch should be of the square root function.