Part III Magma Dynamics – Past Practical Exam Questions

1 A simple 1D steady-state melting column can be described by the following equations:

$$q + v_s = V_0 \tag{1}$$

Equation (1) describes conservation of mass, where q is the Darcy flux, v_s is the solid velocity, and V_0 is the solid upwelling velocity at the base of the column.

$$q = \frac{k_\phi}{\mu}\Delta\rho g \tag{2}$$

Equation (2) is Darcy's law, where k_ϕ is the permeability, μ is the viscosity of the melt, $\Delta\rho$ is the difference in density between solid and melt, and g is acceleration due to gravity.

$$k_\phi = k_0\phi^2 \tag{3}$$

Equation (3) relates permeability to porosity ϕ (the volume fraction of melt), where k_0 is a reference permeability.

$$q = FV_0 \tag{4}$$

Equation (4) relates the Darcy flux to F, the degree of melting.

$$q = \phi v_f \tag{5}$$

Equation (5) relates the Darcy flux to the melt velocity v_f.

$$\frac{\mathrm{d}P}{\mathrm{d}z} = -\rho_s g \tag{6}$$

Equation (6) determines the fluid pressure P, where ρ_s is the density of the solid.

$$C = \frac{\mathrm{d}v_s}{\mathrm{d}z} \tag{7}$$

Equation (7) defines the compaction rate C.

$$\mathcal{P} = \zeta C \tag{8}$$

Equation (8) relates the compaction pressure $\mathcal{P}$ to the compaction rate and the bulk (compaction) viscosity ζ.

$$C_p\frac{\mathrm{d}T}{\mathrm{d}z} = -L\frac{\mathrm{d}F}{\mathrm{d}z} \tag{9}$$

Equation (9) is a statement of conservation of energy, where T is temperature, C_p is heat capacity, and L is latent heat.

In writing these equations it has been assumed that porosities are small ($\phi \ll 1$), that the melt moves much faster than the solid ($v_s \ll v_f$), and that compaction stresses are negligible compared with those due to buoyancy.

The melt productivity $\mathrm{d}F/\mathrm{d}P$ is constant, such that F varies linearly from 0 at the base of the column ($z = 0$) to $F_{\max}$ at the top of the column ($z = h$). Parameter values are given in Table 1.

melt viscosity	μ	10 Pa s
reference permeability	k_0	10^{-8} m^2
density difference	$\Delta\rho$	500 kg m^{-3}
solid mantle density	ρ_s	3300 kg m^{-3}
upwelling velocity at base of column	V_0	30 mm yr^{-1}
height of column	h	60 km
maximum degree of melting	$F_{\max}$	0.2
shear viscosity	η	1.0×10^{19} Pa s
bulk (compaction) viscosity	ζ	1.4×10^{19} Pa s
acceleration due to gravity	g	10 m s^{-2}
specific heat capacity	C_p	1.3×10^{3} J kg^{-1} K^{-1}
latent heat	L	6.4×10^{5} J kg^{-1}
density of ice	ρ_i	1000 kg m^{-3}

Table 1: Parameter values and definitions.

(a) Sketch the variation of $q(z)$, $\phi(z)$, $v_f(z)$, $v_s(z)$, and $\mathcal{P}(z)$ over the melting column. Make sure to label numerical values of these quantities at the top and bottom of the column. [25 minutes]

(b) How does the temperature change from the bottom to the top of the column? [5 minutes]

(c) How long does a chemical signal take to travel from the bottom of the column to the top if it travels at (i) the melt velocity $v_f(z)$; (ii) the solid velocity $v_s(z)$? [10 minutes]

(d) Suppose at the top of the melting column the melt freezes instantly. If compaction stresses are now taken into account, describe qualitatively how the profile of $\phi(z)$ changes in the neighbourhood of the freezing boundary. Over what length-scale would such a change be seen? [10 minutes]

[*Hint: the compaction length δ is defined by*

$$\delta = \sqrt{\frac{k_\phi}{\mu}\left(\zeta + \tfrac{4}{3}\eta\right)}$$

]

(e) Suppose the melting column was originally loaded by a 2 km thick ice layer. If this ice layer were suddenly removed, how would $F(z)$ change? [10 minutes]

2 The following parameter values may be assumed in this question:

Quantity	Symbol	Value
Half-life of ^{230}Th	$\tau_{1/2}^{230\text{Th}}$	75,500 years
Half-life of ^{238}U	$\tau_{1/2}^{238\text{U}}$	4.46×10^{9} years
Half-life of ^{231}Pa	$\tau_{1/2}^{231\text{Pa}}$	32,500 years
Half-life of ^{235}U	$\tau_{1/2}^{235\text{U}}$	7.04×10^{8} years
Half-life of ^{226}Ra	$\tau_{1/2}^{226\text{Ra}}$	1,599 years
Partition coefficient of U	D^{U}	3.4×10^{-3}
Partition coefficient of Th	D^{Th}	1.2×10^{-3}
Partition coefficient of Pa	D^{Pa}	7.0×10^{-5}
Partition coefficient of Ra	D^{Ra}	1.0×10^{-6}
Density of mantle	ρ	3,300 kg m^{-3}
Viscosity of mantle	η	10^{20} Pa s
Acceleration due to gravity	g	9.8 m s^{-2}
Productivity of melting	$\mathrm{d}F/\mathrm{d}P$	10^{-10} Pa^{-1}
Residual porosity during melting	ϕ_0	3×10^{-3}
Effective radius of plume upwelling	a	200 km

^{230}Th is produced by the radioactive decay of ^{238}U; ^{231}Pa by the radioactive decay of ^{235}U; and ^{226}Ra by the radioactive decay of ^{230}Th. The decay constant λ is related to the half-life $\tau_{1/2}$ by

$$\lambda = \frac{\ln 2}{\tau_{1/2}}.$$

The activity of an isotope is the product of its decay constant λ and its concentration c. A daughter/parent activity ratio r is defined by

$$r \equiv \frac{\lambda^d c^d}{\lambda^p c^p}$$

where superscript d refers to the daughter isotope and superscript p refers to the parent.

(a) An activity ratio (^{231}Pa/^{235}U) = 1.25 has been measured in a recent hotspot lava from the Azores. Assuming instantaneous melt extraction, use the following equation and the parameters in the table to estimate $\mathrm{d}F/\mathrm{d}t$, the rate of mantle melting:

$$r = \frac{\phi_0 + D^p + \dfrac{1}{\lambda^d}\dfrac{\mathrm{d}F}{\mathrm{d}t}}{\phi_0 + D^d + \dfrac{1}{\lambda^d}\dfrac{\mathrm{d}F}{\mathrm{d}t}}$$

[5 minutes]

(b) Calculate the expected (^{230}Th/^{238}U) activity ratio for the same lava. [5 minutes]

(c) Assuming a uniform melt productivity dF/dP as given in the table, estimate v_0, the rate of mantle upwelling in mm yr^{-1}. [5 minutes]

(d) Estimate the buoyancy flux Q_B of the mantle plume in Mg s^{-1} from the relationship

$$Q_B = \frac{8\pi\eta}{g} v_0^2.$$

[5 minutes]

(e) The rate of upwelling in a mantle plume decays with distance away from the axis of the plume. A simple expression describing this is

$$v(\rho) = v_0 \exp(-\rho/a)$$

where $v(\rho)$ is the upwelling rate as a function of the distance ρ from the plume axis, a is the effective radius of the upwelling, and v_0 is the upwelling rate at the centre. Assuming the hot spot lava described in (a) was obtained from directly above the plume centre, estimate the (^{231}Pa/^{235}U) and (^{230}Th/^{238}U) activity ratios expected at a distance of 150 km from the plume centre. [10 minutes]

(f) The process of melt ascent is not instantaneous. Suppose it takes 10,000 years for melt to rise to the surface. What would the (^{231}Pa/^{235}U) activity ratio in the melt have been when it first formed if it is 1.25 in the lava now?

Hint: The concentration of the daughter isotope evolves due to radioactive decay as

$$\frac{dc^d}{dt} = -\lambda^d c^d + R$$

where $R \equiv \lambda^p c^p$ is the activity of the parent isotope. R can be treated as a constant in this differential equation because the half-life of the parent isotope is so long.

[10 minutes]

(g) If a measurement of the $\left(^{226}\text{Ra}/^{230}\text{Th}\right)$ activity ratio were made, what additional constraints might it place on melting and melt extraction processes in the plume? [5 minutes]

3 (a) In a laboratory experiment, a sample of partially molten rock is deformed in simple shear for 24 hours until reaching a total shear strain $\gamma = 2$. Estimate the corresponding shear strain rate $\dot{\gamma}$ (in s^{-1}) that the sample has experienced. [4 minutes]

(b) The experimenter records that a shear stress of 50 MPa was applied. Estimate the effective shear viscosity η of the sample. [4 minutes]

(c) The sample has a grain size $d = 10\ \mu$m and a porosity (volume fraction of melt) $\phi = 0.02$. Using the permeability-porosity relation

$$K_\phi = \frac{d^2\phi^2}{C},$$

where $C = 1600$ is a numerical constant, estimate the permeability K_ϕ of the sample. [4 minutes]

(d) Estimate the compaction length δ of the sample, where

$$\delta = \sqrt{\frac{K_\phi\left(\zeta + \frac{4}{3}\eta\right)}{\mu}}.$$

η is the effective shear viscosity of the sample, $\zeta \approx \frac{5}{3}\eta$ is the effective bulk viscosity, and $\mu = 10$ Pa s is the viscosity of the melt. Explain what the compaction length physically represents.

[8 minutes]

(e) A simple model of diffusion creep yields the following relationship between effective shear viscosity η and porosity ϕ,

$$\eta = \eta_0\left(1 - \sqrt{\frac{\phi}{\phi_d}}\right)^2,$$

where η_0 is the shear viscosity in the absence of melt, and $\phi_d = 0.16$ is a constant. Explain why the presence of melt reduces the effective shear viscosity and use the above relationship to calculate the porosity-weakening factor α, defined by

$$\alpha = -\frac{1}{\eta}\frac{\mathrm{d}\eta}{\mathrm{d}\phi}$$

for a porosity $\phi = 0.02$. [10 minutes]

(f) A linear stability analysis for the growth of melt bands under simple shear yields the initial growth rate $\dot{s}$ as

$$\dot{s} = \frac{2\alpha\nu\dot{\gamma}k^2\sin 2\theta}{k^2 + \delta^{-2}}$$

where θ is the angle of the bands to the shear plane. k is the wavenumber of the bands, related to wavelength by

$$k = \frac{2\pi}{\lambda},$$

and ν is given by

$$\nu = \frac{\eta}{\zeta + \frac{4}{3}\eta}.$$

For what angle θ is the growth rate of bands largest? For this band angle, describe the variation of growth rate with wavelength. Estimate the magnitude of the maximum growth rate. Would you expect a significant growth of melt bands over the duration of the experiment? [10 minutes]

(g) What features of melt banding during shear deformation in the laboratory are not well-described by the simple theory that has been assumed here? [5 minutes]

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