Friction Stir Welding

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1 Introduction

Friction Stir Welding is an innovative technique for joining two pieces of metal. A rapidly rotating tool is pushed between two sheets of metal causing friction heating which in turn softens the material. This softening causes a flow around the tool, which leads to greater friction heating. It is the coupling of these two problems, heat and plastic flow, that we would like to understand.

There are several observables that a good model of friction stir welding should be able to predict - the power, the force, the temperature of the tool, and the thickness of the softened region. We are interested in looking at the parameters in the problem which control these observables.

2 Approach

To tackle this problem we have taken a highly idealised approach. Firstly the geometry of the problem is heavily simplified - we consider the tool as a 2D rotating cylinder. To simplify the heat problem we assume we have a steady state. The following laws are assumed to govern the stress and the hardness:

$$\sigma = \kappa(T)\left(\frac{\partial u}{\partial y}\right)^{\alpha}$$

$$\kappa(T) = \kappa_a\left(1 - \frac{T}{T_m}\right)\exp\left(\frac{T_a}{T}\right)$$

Here x and y are locally cartesian co-ordinates on the surface of the cylinder as shown in figure 1. u is the flow velocity (in the x-direction), σ is the stress, κ is the hardness, κ_a is a hardness constant, T is the temperature, T_m is the melting temperature, and T_a is the activation temperature. The tool has radius a and rotates with constant angular velocity Ω.

These are simplified laws governing the stress and the hardness, based on the Norton-Hoff power law formula with a linear dependence on temperature of the shear modulus. More sophisticated formulae are available, such as the Sellars-Tegart Law[1, 3] which involves an additional sinh dependence. Another refinement would be to choose a power law dependence of the shear modulus.

To begin with we just attempt the rotational problem, and later add a slow translation of the tool.

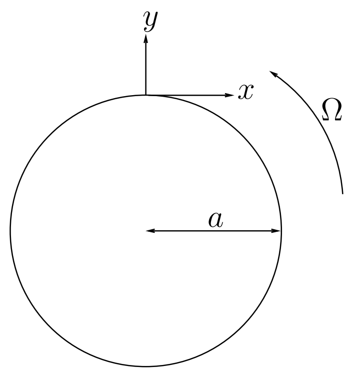

Figure 1: Diagram showing tool with locally cartesian co-ordinates

3 Key ideas

We are most interested in the region in which the metal is significantly softened, where κ is less than twice its minimum value. κ is a sensitive function of T, and so it is possible that there is just a small region δ over which it is significantly soft, where there is only a small change in temperature ΔT from its maximum value T_{max}. So ΔT satisfies:

$$\kappa(T_{max} - \Delta T) = 2\kappa(T_{max})$$

Another key idea is one of conservation of heat. The heat generated must be the same as the heat lost.

$$\text{heat generated} = \text{thickness} \times \text{stress} \times \text{shear rate}$$

$$= \delta \times \sigma \times \frac{\Omega a}{\delta} = \kappa(T_{max})\delta\left(\frac{\Omega a}{\delta}\right)^{1+\alpha} \tag{1}$$

$$\text{heat loss} = \text{thermal conductivity} \times \text{temperature gradient to `infinity'}$$

$$= k \times \frac{T_{max} - T_{\infty}}{a \log R/a}^{1} \tag{2}$$

Note we have another expression for the heat loss given by:

$$\text{heat loss} = \text{heat conducted out of thin softened region}$$

$$= \frac{k\Delta T}{\delta} \tag{3}$$

It is not possible to achieve this balance at moderate temperatures. For example, suppose we have a thick layer so that $a = \delta$. Consider the case of aluminium, for which example data is given in appendix A. Then at a moderate temperature of $T = 400$K, 6.69×10^7J of heat is produced, but only 1.57×10^6J lost to the surroundings. Thus the temperature around the tool rises causing further softening. This has the effect of reducing the heat generated while increasing the heat lost. Hence there is a possibility of finding a balance between the two factors at higher temperatures.

To simplify looking at the effect of hardness we break down the hardness law into two regimes - the exponential and the linear. For hard metals such as titanium the linear factor is dominant in governing the behavior of $\kappa(T)$ whereas for softer metals such as aluminium the exponential factor is more relevant.

[1] The reason for the log factor is described in section 6.

4 Achieving balance in the exponential regime

Suppose we are in the situation where the exponential factor of hardness is dominant so that:

$$\kappa(T) = \kappa_a \exp\left(\frac{T_a}{T}\right)$$

So

$$\kappa(T_{max} - \Delta T) \approx \kappa_a \exp\left(\frac{T_a}{T_{max}}\right) \exp\left(\frac{T_a \Delta T}{{T_{max}}^2}\right)$$

For this to approximately double we need:

$$\frac{T_a \Delta T}{{T_{max}}^2} = 1 \tag{4}$$

Assuming $T_{max} \gg T_\infty$ equations 1, 2, 3, and 4 imply:

$$e^{\left(\frac{T_a}{T_{max}}\right)} \left(\frac{T_a}{T_{max}}\right)^{1+\alpha} = \frac{kT_a}{\kappa_a (\Omega a)^{1+\alpha} (a \log R/a)^{1-\alpha}}$$

This can be solved numerically to give T_{max}. Then δ is given by:

$$\delta = a \log R/a \frac{T_{max}}{T_a}$$

For aluminium the above give $T_{max} = 697\text{K}, \delta = 0.932a = 0.00466\text{m}, \Delta T = 217\text{K}$. Note that this is quite a thick boundary layer.

5 Achieving balance in the linear regime

Now suppose that the linear factor is dominant so:

$$\kappa(T) = \kappa_m \left(1 - \frac{T}{T_m}\right)$$

Where κ_m is given by:

$$\kappa_m = \kappa_a e^{\frac{T_a}{T_m}}$$

Now:

$$\kappa(T_{max} - \Delta T) \approx \kappa_m \left(1 - \frac{T}{T_m}\right)\left(1 + \frac{\Delta T}{T_m - T_{max}}\right)$$

Hence for this to approximately double we need:

$$\Delta T = T_m - T_{max}$$

Using equations 2 and 3, and again assuming $T_{max} \gg T_\infty$ we get:

$$\delta = \frac{T_m - T_{max}}{T_{max}} a \log R/a \approx \frac{T_m - T_{max}}{T_m} a \log R/a$$

Now using equation 1 we see that:

$$\delta = \left(\frac{kT_m}{\kappa_m(\Omega a)^{1+\alpha}}\right)^{\frac{1}{1-\alpha}} = a\left(\frac{kT_m}{\kappa_m(\Omega)^{1+\alpha}a^2}\right)^{\frac{1}{1-\alpha}}$$

For titanium the above give $T_{max} = 1906\text{K}, \delta = 0.0412a = 2.06 \times 10^{-4}\text{m}, \Delta T = 27\text{K}$.

6 Boundary layer theory in the exponential case

We assume we have a thin layer, and as such we assume u and T are functions of y only. The governing equations are:

$$\frac{\partial \sigma}{\partial y} = 0 \text{ (momentum)}$$

$$k\frac{\partial^2 T}{\partial y^2} + \sigma\frac{\partial u}{\partial y} = 0 \text{ (heat)}$$

In the heat equation we assume the thermal conductivity of the metal k remains constant.

Let $\eta = \frac{y}{\delta}$, $u = \Omega a f(\eta)$, $T = T_{max} - \Delta T g$. Then these equations non-dimensionalise to become:

$$e^g f'^{\alpha} = S \tag{5}$$

$$g'' - e^g f'^{1+\alpha} = 0 \tag{6}$$

Outside the boundary layer we assume we just have diffusion of heat, given by $\nabla^2 T = 0$. In cylindrical co-ordinates the solution of this equation consists of a logarithm. To fix the problems associated with this we impose that at some fixed distance R that $T = T_\infty$. Choice of R is discussed in section 13. On the tool we assume that $T = T_{max}$. Hence outside the boundary layer the solution of the heat equation is:

$$T(r) = T_\infty + (T_{max} - T_\infty)\frac{\log R/r}{\log R/a}$$

Hence the temperature gradient is:

$$\frac{\partial T}{\partial r} = -\frac{T_{max} - T_\infty}{r \log R/a}$$

Evaluating this on $r = a$ accounts for the log factor found earlier in the energy balance. On the tool we must have $u = -\Omega a$, and at large distances we must have $u = 0$. Hence the boundary conditions for f are $f(0) = -1$, $f(\infty) = 0$. We assume no heat flow through the tool, and so $g'(0) = 0$. To match the temperature gradient calculated above, we must have $g'(\infty) = 1$.

Using 5, 6 simplifies to:

$$g'' - Sf' = 0$$

Integrating this we see from the boundary conditions that $S = 1$.

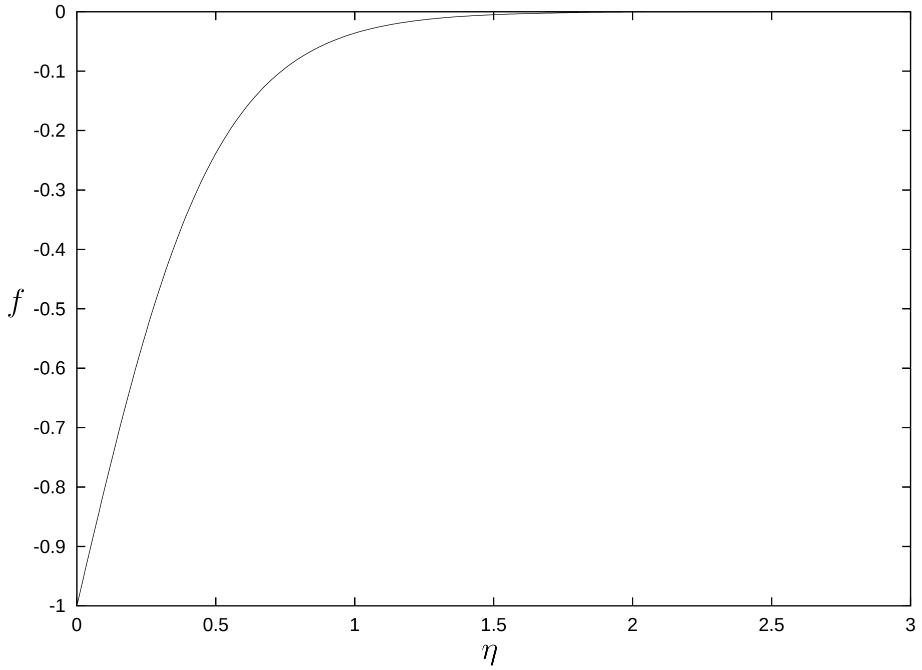

Figure 2: f profile for the exponential boundary layer

So the equations become:

$$f' = e^{-\frac{g}{\alpha}}$$

$$g'' = f'$$

These equations have a first integral:

$$g'^2 = 2\alpha\left(e^{-g(0)/\alpha} - e^{-g/\alpha}\right)$$

Examining the boundary conditions we see this implies $g(0) = \alpha \log 2\alpha$.
In the case of $\alpha = 0.25$, $g(0) = -0.17329$, and hence for aluminium $T_{tool} = 735\text{K}$
The profiles from numerically solving these equations can be seen in figures 2 and 3.
The constant stress is given from the expression:

$$\sigma = \kappa_a e^{\frac{T_a}{T_{max}}} \left(\frac{\Omega a}{\delta}\right)^{\alpha}$$

In the case of aluminium this is 4.10×10^7Pa Hence the couple on the tool $G = 2\pi a \times \sigma \times a = 6450\text{Pam}^2$. Thus the rate of working is $\Omega G = 1.9 \times 10^5\text{Wm}^{-1}$. So for a tool of length $l = 0.02$m the power output is 3.8kW.

7 Boundary layer theory in the linear case

In the case of the linear law we let $\eta = \frac{y}{\delta}$, $u = \Omega a f(\eta)$, $T = T_m - \Delta T g$. Then these equations non-dimensionalise to become:

$$g f'^{\alpha} = S \tag{7}$$

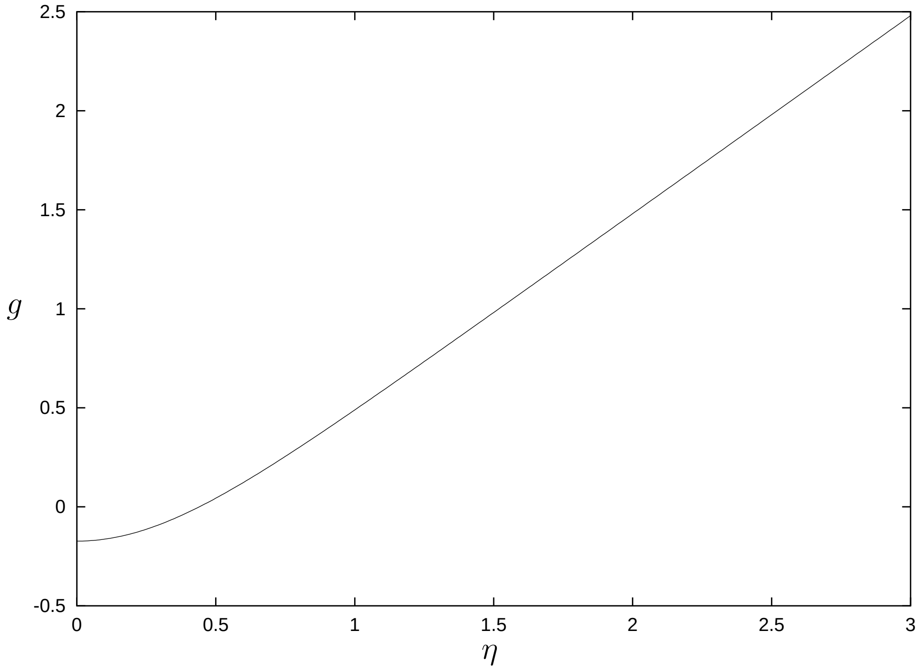

Figure 3: g profile for the exponential boundary layer

$$g'' - g{f'}^{1+\alpha} = 0 \tag{8}$$

Again the boundary conditions are $f(0) = -1$, $f(\infty) = 0$, $g'(0) = 0$, $g'(\infty) = 1$.
Using 7, 8 simplifies to:

$$g'' - Sf' = 0$$

Integrating this we again see from the boundary conditions that $S = 1$.
So the equations become:

$$f' = g^{-1/\alpha}$$

$$g'' = f'$$

These equations have a first integral:

$$g'^2 = \frac{2\alpha}{1-\alpha}\left(g(0)^{1-1/\alpha} - g^{1-1/\alpha}\right)$$

Examining the boundary conditions we see this implies $g(0) = \left(\frac{2\alpha}{1-\alpha}\right)^{\frac{\alpha}{1-\alpha}}$.
In the case of $\alpha = 0.25$, $g(0) = 0.87358$, and hence for titanium $T_{tool} = 1909\text{K}$.
The profiles from numerically solving these equations can be seen in figures 4 and 5.
The constant stress is given from the expression:

$$\sigma = \kappa_m \frac{\Delta T}{T_m}\left(\frac{\Omega a}{\delta}\right)^{\alpha}$$

In the case of titanium this is $1.89 \times 10^7\text{Pa}$. Hence the couple on the tool $G = 2\pi a \times \sigma \times a = 3000\text{Pam}^2$. Thus the rate of working is $\Omega G = 89000\text{Wm}^{-1}$. So for a tool of length $l = 0.02\text{m}$ the power output is 1.8kW. The reason that the power output is less for titanium can be seen when considering an alternate expression for the power

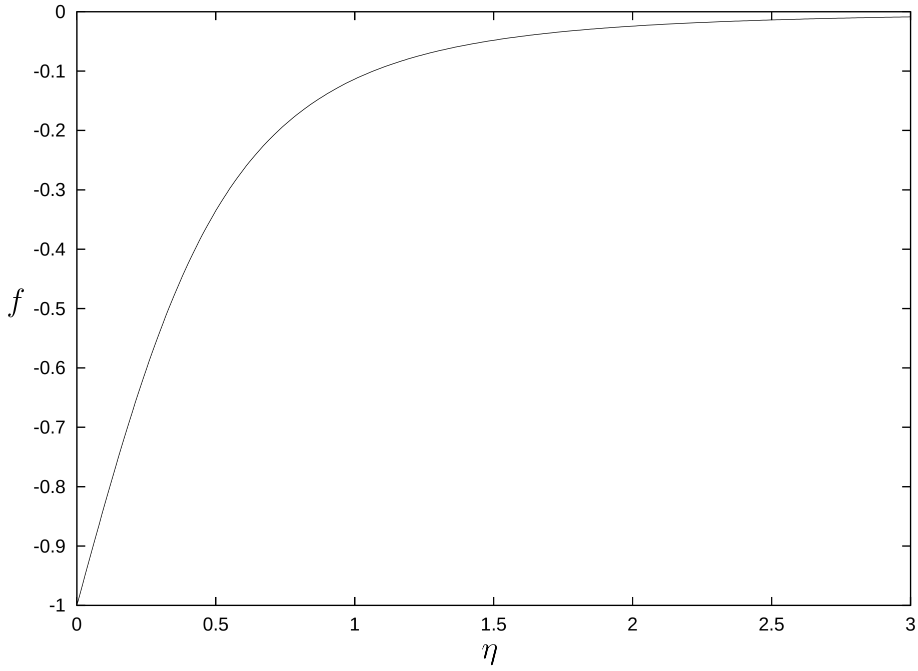

Figure 4: f profile for the linear boundary layer

generated given simply by looking at the heat loss in equation 2. Although T_{max} is higher for titanium, the thermal conductivity k is much lower and thus the power output is less.

8 The full tensorial equations

We now drop the assumption of a thin layer. Consider the general problem where in cylindrical polar co-ordinates we wish to find $u_r(r, \theta)$, $u_\theta(r, \theta)$, $T(r, \theta)$. The mechanical problem is given by:

$$\sigma_{ij} = 2\kappa(T)\gamma^{\alpha-1}e_{ij} \text{ where } \gamma^2 = 2e_{ij}e_{ij}$$

$$\nabla.\sigma = \nabla p$$

$$\nabla.\mathbf{u} = 0$$

where σ_{ij}, e_{ij} are the stress[2] and strain tensors respectively.
The full heat equation is:

$$\rho c_p \mathbf{u}.\nabla T = k\nabla^2 T + \sigma_{ij}e_{ij}$$

We neglect the $\rho c_p \mathbf{u}.\nabla T$ term (discussed in section 13).

[2]Note that here we are using the deviatoric stress tensor for convenience of notation.

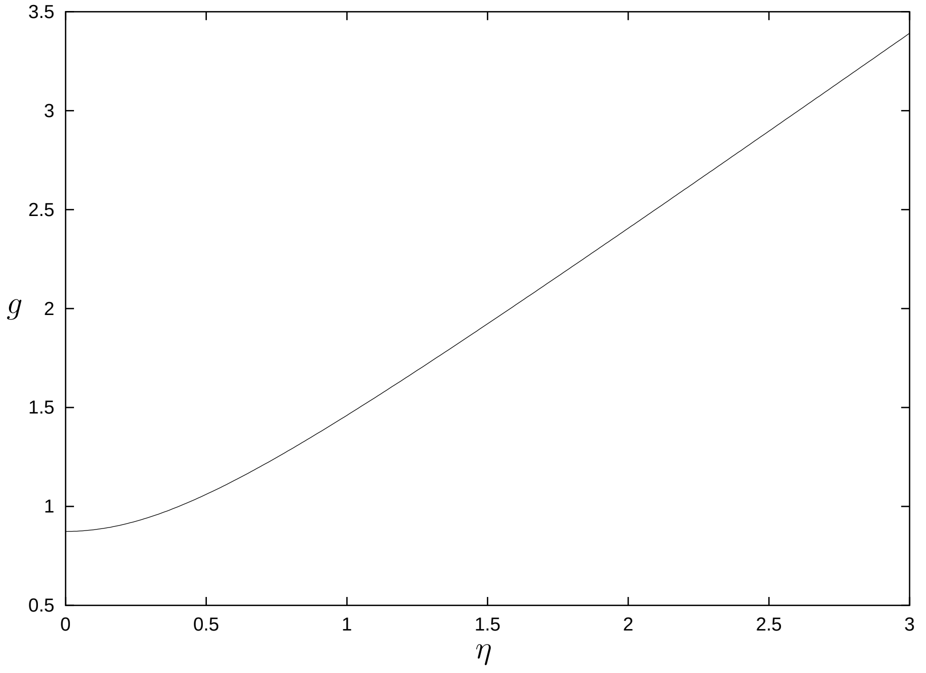

Figure 5: g profile for the linear boundary layer

9 Non-boundary layer rotational problem

For the rotational problem the full tensorial equations can be simplified. Here we only have $u_\theta(r)$, $T(r)$. Only the tangential components of stress and strain, $\sigma_{r\theta}$, $e_{r\theta}$ are non-zero. The stress relationship becomes:

$$\sigma_{r\theta} = \kappa(T)(2e_{r\theta})^\alpha$$

$$\nabla.\sigma = 0$$

and thus the heat equation becomes:

$$k\nabla^2 T + \kappa(T)(2e_{r\theta})^{1+\alpha} = 0$$

To non-dimensionalise these equations scale r with a, T with T_m, κ with κ_a, u_θ with Ωa, $e_{r\theta}$ with Ω. Write $2e_{r\theta} = h$, $u_\theta = u$. Introduce the non-dimensional groups:

$$K = \frac{\kappa_a \Omega^{1+\alpha} a^2}{kT_m}$$

$$A = \frac{T_a}{T_m}$$

In non-dimensional form the equations become:

$$h = r\frac{\partial}{\partial r}\left(\frac{u}{r}\right)$$

$$\kappa = (1-T)e^{A/T}$$

The momentum equation implies that:

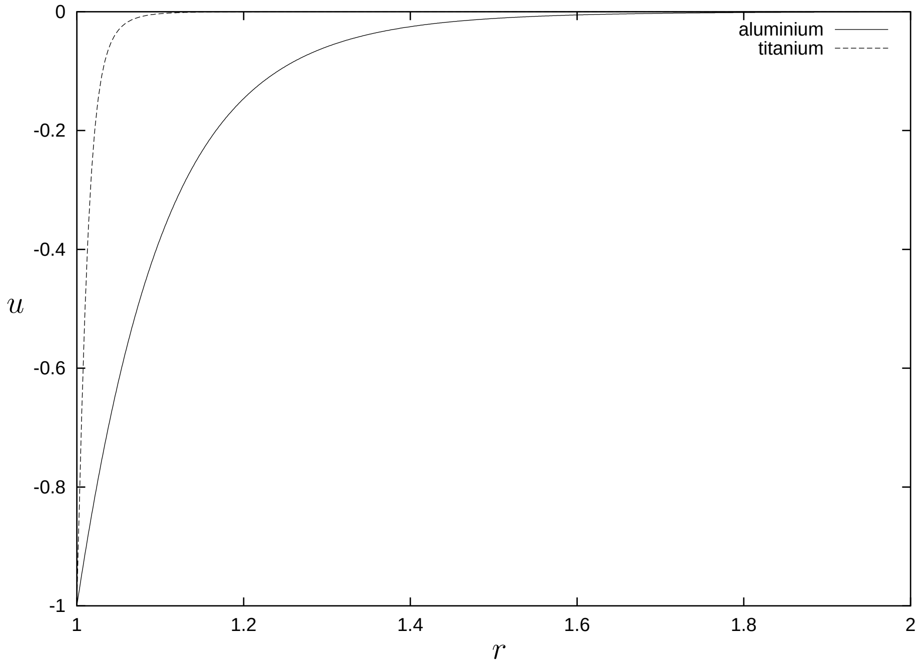

Figure 6: u profiles comparing aluminium and titanium

$$r^2 \kappa h^\alpha = S$$

The heat equation becomes:

$$\frac{1}{r}\frac{\partial}{\partial r}(rT') + K\kappa h^{1+\alpha} = 0$$

Then these equations simplify to:

$$T'' = -\frac{T'}{r} - K\frac{S}{r^2}\left(\frac{S}{r^2\kappa}\right)^{1/\alpha}$$

$$u' = \frac{u}{r} + \left(\frac{S}{r^2\kappa}\right)^{1/\alpha}$$

The boundary conditions are given by $f(1) = -1$, $f(\infty) = 0$, $T'(1) = 0$, $T(R) = T_\infty/T_m$.

Profiles from solving these equations numerically are given in figures 6, 7 and 8.

Using non-boundary layer theory we find $T_{tool} = 672$K for aluminium, and $T_{tool} = 1916$K for titanium. The corresponding boundary layer thicknesses are 0.0049m and 2.19×10^{-4}m. These results are close to the boundary layer results.

In dimensional terms the stress on the tool is given by:

$$\sigma = \kappa_a \Omega^\alpha S$$

For aluminium this is 4.03×10^7Pa. Following the calculations done earlier, this again leads to a power output of 3.8kW. For titanium the stress is 1.59×10^7Pa. The leads to a power output of 1.5kW, a slightly smaller result than the boundary layer analysis.

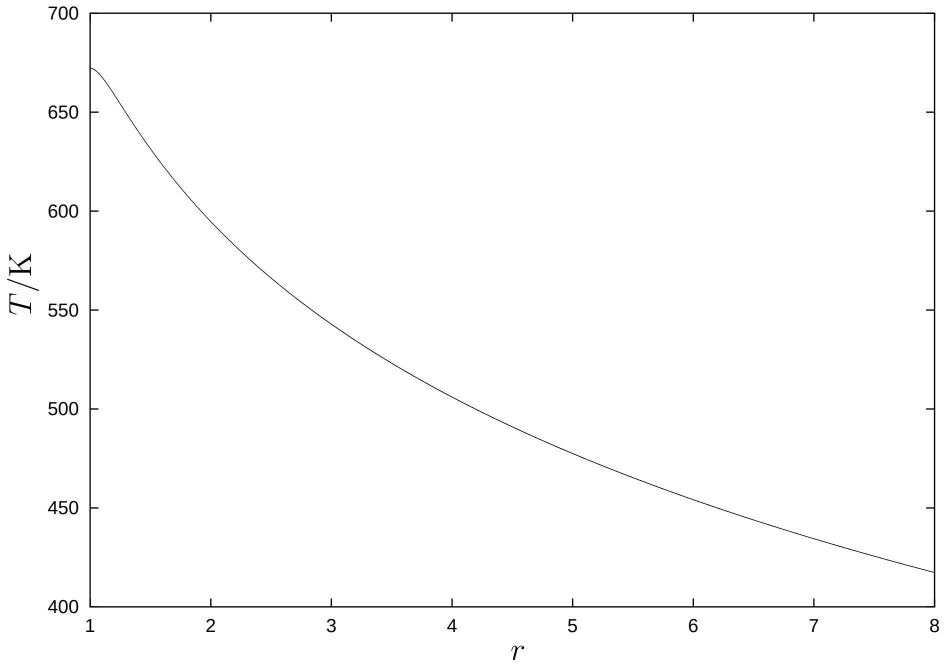

Figure 7: Temperature profile for aluminium

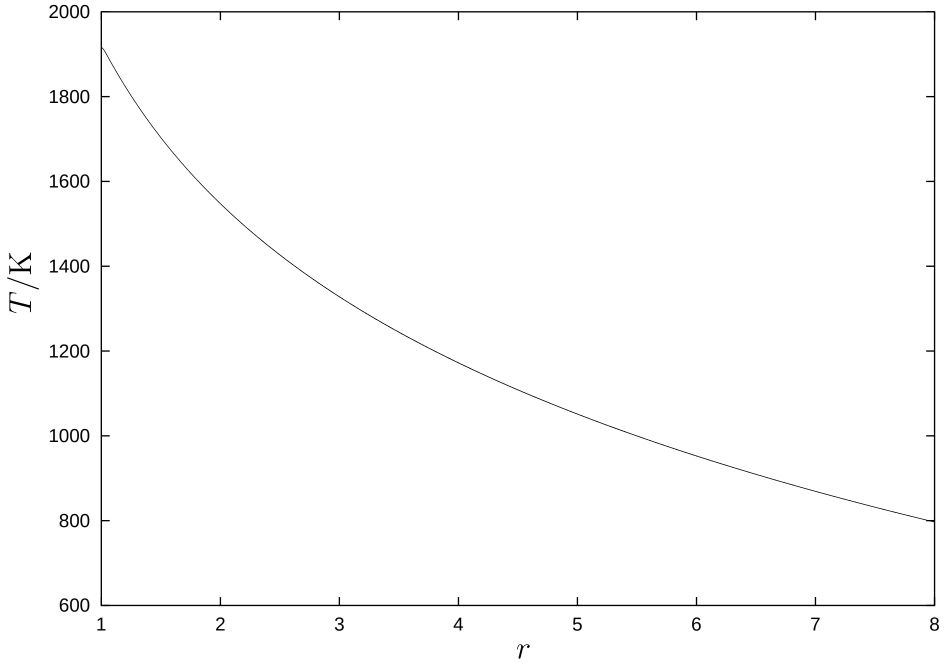

Figure 8: Temperature profile for titanium

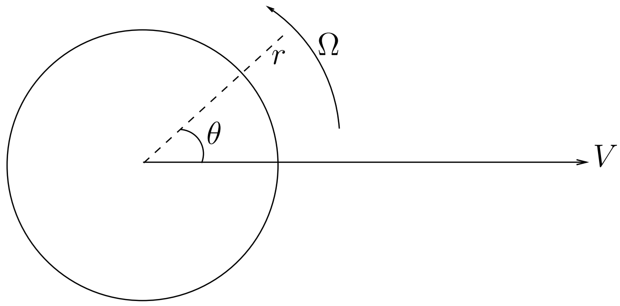

Figure 9: Diagram showing translating tool with cylindrical polar co-ordinates

10 The translational problem

We now let the tool move at a small constant velocity V such that $\frac{V}{\Omega a} = \epsilon \ll 1$. Take cylindrical polar co-ordinates as drawn in figure 9. For the numerical examples this gives $\epsilon = 0.0067$. We treat the translational problem as a small perturbation to the rotational problem. We must now use the full tensorial form of the equations. Proceed with the same non-dimensionalisation as before. Let:

$$u_r = -\epsilon \cos\theta \frac{\psi_1(r)}{r}$$

$$u_\theta = u\left(r\right) + \epsilon \sin\theta {\psi_1}'\left(r\right)$$

By writing the velocity in this form we automatically ensure $\nabla.\mathbf{u} = 0$. The components of the strain tensor e_{ij} are then given by:

$$e_{rr} = -\epsilon \cos\theta \frac{\partial}{\partial r}\left(\frac{\psi_1}{r}\right)$$

$$e_{r\theta} = \frac{1}{2}\left(r\frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \epsilon \sin\theta \left(r\frac{\partial}{\partial r}\left(\frac{{\psi_1}'}{r}\right) + \frac{\psi_1}{r^2}\right)\right)$$

$$e_{\theta\theta} = \epsilon \cos\theta \frac{\partial}{\partial r}\left(\frac{\psi_1}{r}\right) = -e_{rr}$$

To first order in ϵ we have that:

$$\gamma = r\frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \epsilon \sin\theta \left(r\frac{\partial}{\partial r}\left(\frac{{\psi_1}'}{r}\right) + \frac{\psi_1}{r^2}\right) = 2e_{r\theta}$$

Define h, $f_{r\theta}$ so that $e_{r\theta} = \frac{h}{2} + \epsilon \sin\theta f_{r\theta}$. Similarly define $e_{rr} = \epsilon \cos\theta f_{rr}$. So:

$$h = r\frac{\partial}{\partial r}\left(\frac{u}{r}\right)$$

as before and:

$$f_{r\theta} = \frac{1}{2}\left(r\frac{\partial}{\partial r}\left(\frac{{\psi_1}'}{r}\right) + \frac{\psi_1}{r^2}\right)$$

$$f_{rr} = -\frac{\partial}{\partial r}\left(\frac{\psi_1}{r}\right)$$

Let $T(r,\theta) = T(r) + T_1(r)\epsilon \sin\theta$ and $\kappa(T(r,\theta)) = \kappa(r) + \kappa_1(r)\epsilon \sin\theta$.
Then:

$$\kappa = (1 - T)e^{A/T}$$

as before and:

$$\kappa_1 = -T_1 e^{A/T}\left(1 + \frac{A(1-T)}{T^2}\right)$$

To first order in ϵ we can write the components of the stress tensor as:

$$\sigma_{rr} = \kappa h^{\alpha-1}\left(2f_{rr}\right)\epsilon\cos\theta$$

$$\sigma_{r\theta} = \kappa h^{\alpha} + \epsilon\sin\theta\left(\kappa_1 h^{\alpha} + \alpha\kappa h^{\alpha-1}\left(2f_{r\theta}\right)\right)$$

$$\sigma_{\theta\theta} = -\kappa h^{\alpha-1}\left(2f_{rr}\right)\epsilon\cos\theta = -\sigma_{rr}$$

In cylindrical polar co-ordinates the momentum equation $\nabla.\sigma = \nabla p$ is:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{1}{r}\frac{\partial}{\partial\theta}(\sigma_{\theta r}) - \frac{\sigma_{\theta\theta}}{r} = \frac{\partial p}{\partial r}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{r\theta}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}(\sigma_{\theta\theta}) = \frac{1}{r}\frac{\partial p}{\partial\theta}$$

Define $p(r,\theta) = \epsilon p(r)\cos\theta$. Then equating terms of order ϵ^0 gives the same governing equations as in the previous section. Let $s_{r\theta}$ and s_{rr} be the order ϵ components of σ. Then the momentum equation becomes:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 s_{rr}\right) + \frac{s_{\theta r}}{r} = p'$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 s_{r\theta}\right) + \frac{s_{rr}}{r} = -\frac{p}{r}$$

The heat equation for the order ϵ term is:

$$\frac{1}{r}\frac{\partial}{\partial r}(rT_1{}') - \frac{T_1}{r^2} + K\left(\kappa_1 h^{1+\alpha} + (1+\alpha)\kappa h^{\alpha}\left(2f_{r\theta}\right)\right) = 0$$

To solve these equations we work in terms of the variables ψ_1, ψ_1', $s_{r\theta}$, $(s_{rr} - p)$, T_1, T_1'. Then we calculate:

$$\mu = \kappa h^{\alpha-1}$$

$$f_{rr} = \frac{\psi_1}{r^2} - \frac{\psi_1{}'}{r}$$

$$f_{r\theta} = \frac{s_{r\theta} - \kappa_1 h^{\alpha}}{2\mu\alpha}$$

$$s_{rr} = 2\mu f_{rr}$$

$$p = s_{rr} - (s_{rr} - p)$$

Then the governing differential equations can be expressed in the form:

$$\frac{\partial}{\partial r}(\psi_1) = \psi_1'$$

$$\frac{\partial}{\partial r}(\psi_1') = 2f_{r\theta} - f_{rr}$$

$$\frac{\partial}{\partial r}(s_{r\theta}) = -\frac{2s_{r\theta} + s_{rr} + p}{r}$$

$$\frac{\partial}{\partial r}(s_{rr} - p) = -\frac{2s_{rr} + s_{r\theta}}{r}$$

$$\frac{\partial}{\partial r}(T_1) = T_1'$$

$$\frac{\partial}{\partial r}(T_1') = -\frac{T_1'}{r} + \frac{T_1}{r^2} - K\left(\kappa_1 h^{1+\alpha} + (1+\alpha)\kappa h^{\alpha}\left(2f_{r\theta}\right)\right)$$

On the tool we have $\psi_1(1) = 0$, $\psi_1'(1) = 0$ since the tool rotates at constant velocity. At infinity we just have a uniform flow. Thus we demand that $\psi_1(r) = r$ at large distances. Hence we demand $r\psi_1' - \psi_1 \to 0$ and $\psi_1' \to 1$ as $r \to \infty$. [A more sophisticated examination of the boundary condition at infinity has been looked at. The terms in solving for ψ_1 at large distances are thought to go like r to the powers of $\left(3, 1, \frac{3\alpha-2}{\alpha}, \frac{\alpha-2}{\alpha}\right)$, neglecting log factors. Thus the condition at infinity can be chosen to be such that the coefficient of the r^3 term is zero, and the r term 1. However, we have found the simplified conditions to be suitable for our purposes.]

As before we want no heat flow out of the tool, so set $T_1'(1) = 0$. At large distances we have pure conduction of heat so we have:

$$\nabla^2\left(T_1(r)\cos\theta\right) = 0$$

This has solutions of the form $T_1(r) = r, \frac{1}{r}$. Hence our boundary condition at infinity should ensure the $\frac{1}{r}$ behavior is taken, and so impose the condition $\frac{T_1}{r} + T_1' \to 0$ as $r \to \infty$.

We are most interested in the force on the tool. In dimensional terms this is given by:

$$\mathbf{F} = \int_{\text{tool}} \left(-p\mathbf{n} + \sigma.\mathbf{n}\right) dS = -\epsilon\pi\kappa_a\Omega^{\alpha} al\left(s_{r\theta} - (s_{rr} - p)\right)\mathbf{i}$$

Where $\mathbf{i}$ is the unit vector in the direction of motion i.e. the force is in the opposite direction to the motion of the tool. Here $s_{r\theta}$, s_{rr}, and p refer to the values on the tool. The results from numerically solving these equations give a force of 2.0kN for aluminium, and 25kN for titanium. This leads to a translational power output of only 2W and 25W respectively for the translational motion. Only a very small change in the heat field is observed - on the tool this is 0.9K for aluminium and 1.0K for titanium. For these examples it is certainly the case that the rotational motion is the dominant cause of the heating. Note there is quite a difference between the perturbed temperature profiles of aluminium and titanium (figures 12 and 13) although the effects of both are very small. As far as the coupling of the problems is concerned the dominant effect seems to be one way - the increasing shear rates cause heating, but the effect of this heating on softening the material and causing greater shear rates is small.

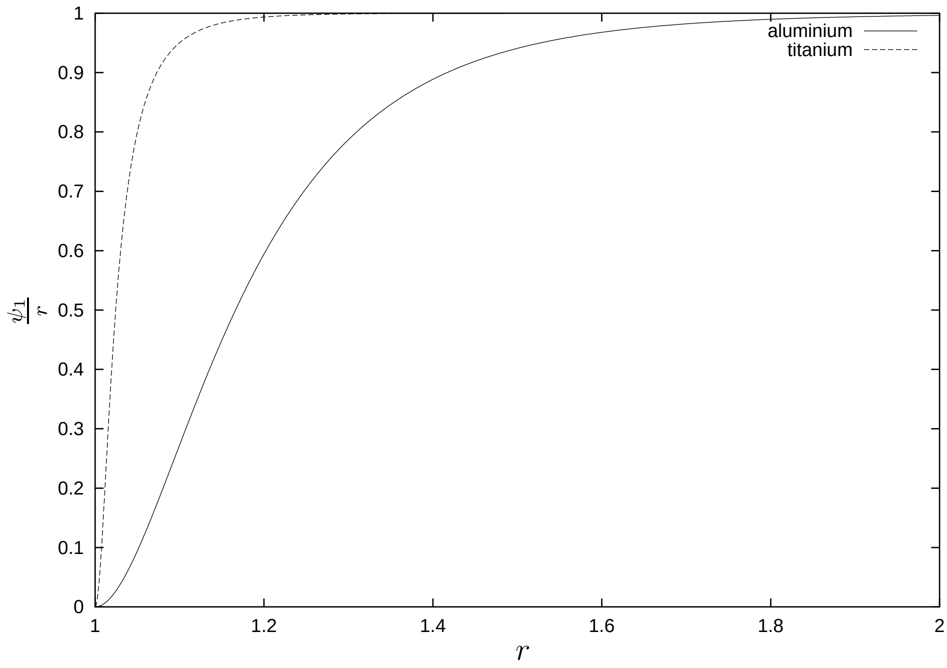

Figure 10: Radial velocity profile of perturbed state

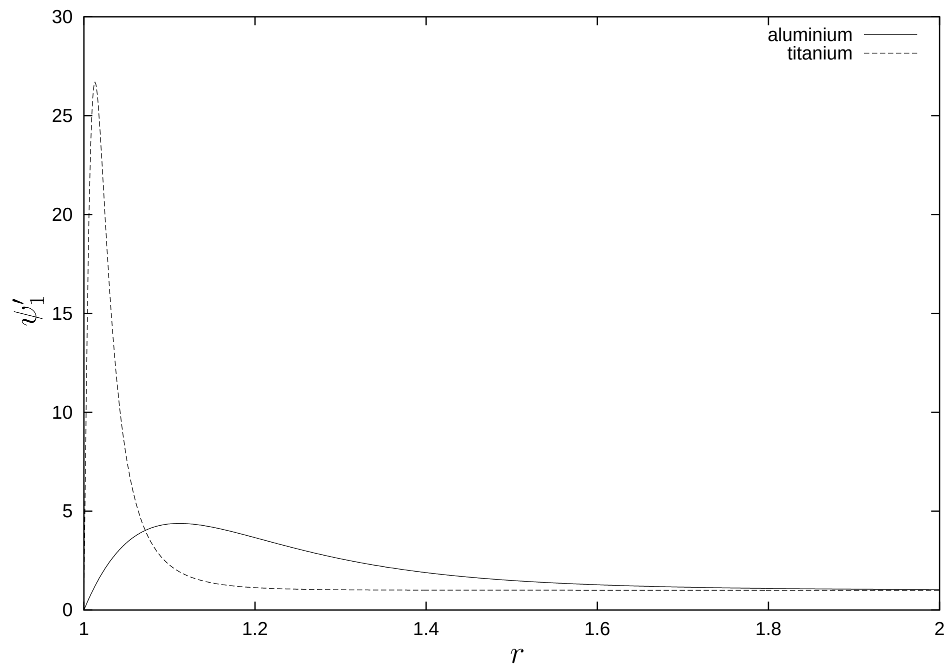

Figure 11: Tangential velocity profile of perturbed state

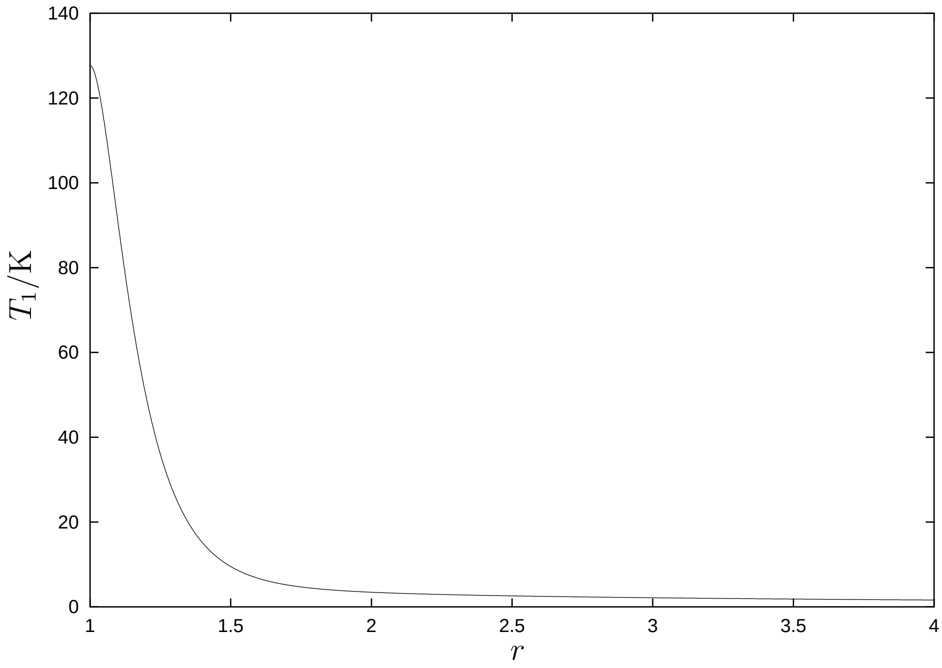

Figure 12: Temperature profile of perturbed state for aluminium

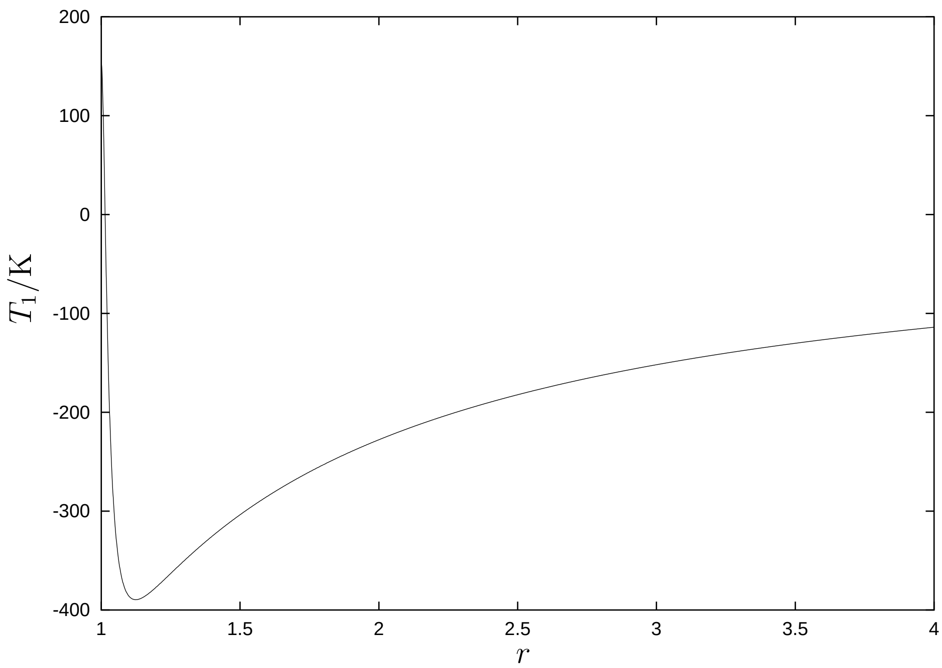

Figure 13: Temperature profile of perturbed state for titanium

11 Boundary layer analysis of the linear translational problem

Since in the case of a linear law we have a thin boundary layer ($\delta = 0.04a$) it seems reasonable to expect a boundary layer analysis of the translation problem to produce the same results more easily. We begin as before letting $\eta = \frac{y}{\delta}$. Now let

$$\epsilon = \frac{V}{\Omega a}\frac{a}{\delta}$$

$$u = \Omega a\left(f(\eta) + \epsilon f_1(\eta)\sin\theta\right)$$

$$p = \kappa_m \frac{\Delta T}{T_m}\frac{a}{\delta}\left(\frac{\Omega a}{\delta}\right)^{\alpha} \epsilon p_1 \cos\theta$$

$$T = T_m - \Delta T\left(g(\eta) + \epsilon g_1(\eta)\sin\theta\right)$$

Note that ϵ is defined differently to before, and now includes a geometry ratio. For the example data we have $\epsilon = 0.007$ for aluminium and $\epsilon = 0.16$ for titanium. Here we assume p_1 to be constant. In the boundary layer the momentum equation takes the form:

$$-\frac{\partial p}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

which equating order ϵ terms implies:

$$p_1 + \left(g_1 f'^{\alpha} + \alpha g f'^{\alpha-1} f_1'\right)' = 0$$

This can be integrated to give

$$p_1\eta + S_1 + g_1 f'^{\alpha} + \alpha g f'^{\alpha-1} f_1' = 0$$

The heat equation implies:

$$g_1'' - \left(g_1 f'^{\alpha+1} + (\alpha+1) g f'^{\alpha} f_1'\right) = 0$$

Using $gf'^{\alpha} = 1$ these can be rewritten as

$$f_1' = -\frac{f'}{\alpha}\left(p_1\eta + S_1 + \frac{g_1}{g}\right)$$

$$g_1'' = \frac{g_1}{g} f' + (\alpha+1) f_1'$$

The boundary conditions on g_1 are chosen to be $g_1'(0) = 0$, $g_1'(\infty) = 0$, i.e. no heat flow through the tool, no heat loss to infinity. For f_1 we impose that $f_1(0) = 0$, $f_1(\infty) = 0$ and additionally:

$$\int_0^{\infty} f_1(\eta)\,d\eta = 1$$

To see where this comes from consider the conservation of flux of material around the tool. For a uniform flow we must have that:

$$\text{flux } q = q_0 + \epsilon V a \sin\theta = \int_0^\infty u dy$$

Equating the order ϵ terms gives the above condition.

Solving these equations numerically gives values of $p_1 = 0.431$, $S_1 = -0.214$ and $g_1(0) = -0.161$. Again we are most interested in the magnitude of force on the tool. This is given by:

$$F = \epsilon \pi a l \kappa_m \frac{\Delta T}{T_m} \frac{a}{\delta} \left(\frac{\Omega a}{\delta} \right)^\alpha (p_1 - S_1)$$

For our results this gives 15kN, which is just over half the value given by the non-boundary layer theory. Given the small thickness of the boundary layer ($0.04a$) it is striking that there is such a discrepancy when comparing with the non-boundary layer theory. This can be explained by considering the shear rates involved. For the boundary layer we have:

$$\gamma = f' = \left(\frac{1}{\kappa} \right)^{\frac{1}{\alpha}}$$

whereas for the non-boundary layer theory we have:

$$\gamma = h = \left(\frac{S}{r^2 \kappa} \right)^{\frac{1}{\alpha}}$$

The only essential difference between these two expressions is the presence of an $r^{(2/\alpha)}$ factor. However since α is small, in this case $\alpha = 0.25$, this is a high power of r, here r^8. Looking at the flow profiles we see that most of the flow is confined within a region of thickness 0.1a. Now $(1.1)^8 \approx 2.1$, and so due the large powers involved this difference is a significant one. Although numerically the boundary theory may give quite different answers, it exhibits a general similarity in structure to the non-boundary layer theory, as can be seen when examining the graphs in figures 14 and 15.

12 Boundary layer analysis of the exponential translational problem

The boundary layer thickness in the exponential case is very thick ($0.93a$) which suggests a boundary layer analysis will be even worse in this case. However it is worthwhile to examine it. Proceed as before with $\eta = \frac{y}{\delta}$. Now let

$$\epsilon = \frac{V}{\Omega a} \frac{a}{\delta}$$

$$u = \Omega a \left(f(\eta) + \epsilon f_1(\eta) \sin\theta \right)$$

$$p = \kappa_a e^{\frac{T_a}{T_{max}}} \left(\frac{\Omega a}{\delta} \right)^\alpha \epsilon p_1 \cos\theta$$

$$T = T_{max} - \Delta T \left(g(\eta) + \epsilon g_1(\eta) \sin\theta \right)$$

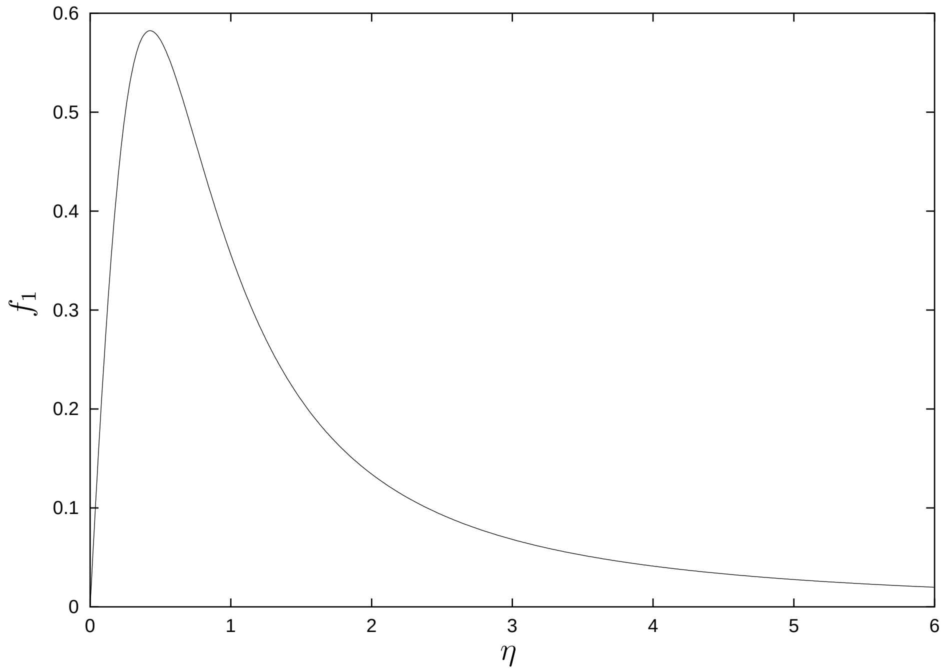

Figure 14: f_1 profile for the linear boundary layer

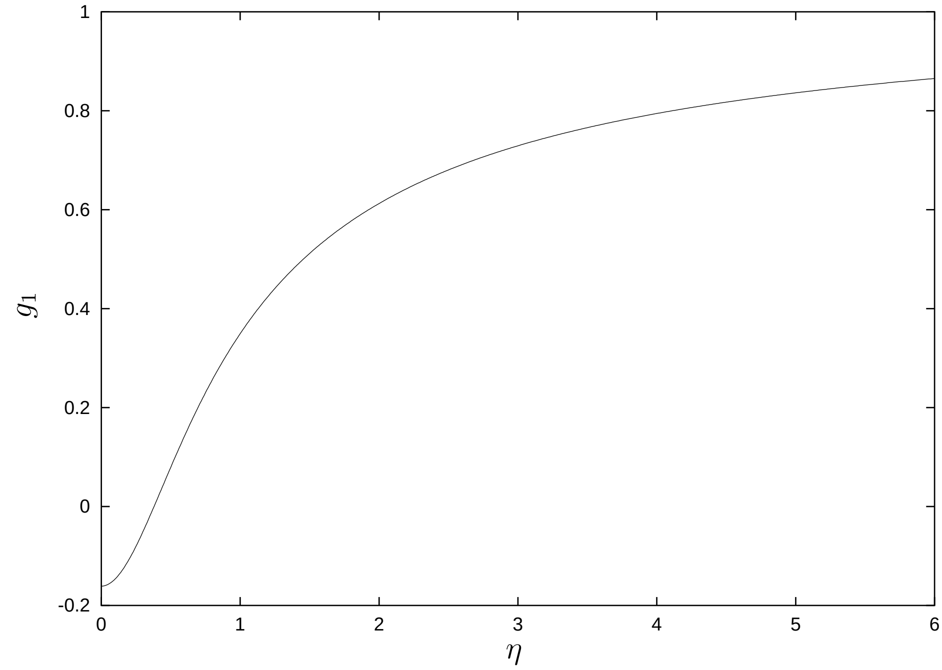

Figure 15: g_1 profile for the linear boundary layer

Again assume p_1 to be constant. The hardness is given to first order in ϵ by:

$$\kappa(T) = \kappa_a e^{\frac{T_{max}}{T_a}} e^{g} \left(1 + \epsilon g_1 \sin\theta\right) = \kappa_a e^{\frac{T_{max}}{T_a}} \left(\kappa + \epsilon \kappa_1 \sin\theta\right)$$

The momentum equation implies:

$$p_1 + \left(\kappa_1 f'^{\alpha} + \alpha\kappa f'^{\alpha-1} f_1'\right)' = 0$$

This can be integrated to give

$$p_1\eta + S_1 + \kappa_1 f'^{\alpha} + \alpha\kappa f'^{\alpha-1} f_1' = 0$$

The heat equation implies:

$$g_1'' - \left(\kappa_1 f'^{\alpha+1} + (\alpha+1)\kappa f'^{\alpha} f_1'\right) = 0$$

Using $\kappa f'^{\alpha} = 1$ these can be rewritten as

$$f_1' = -\frac{f'}{\alpha}\left(p_1\eta + S_1 + g_1\right)$$

$$g_1'' = g_1 f' + (\alpha+1) f_1'$$

We impose the same boundary conditions as before.

Solving these equations numerically gives values of $p_1 = 1.853$, $S_1 = -0.642$ and $g_1(0) = -0.355$. The force is given by:

$$F = \epsilon\pi a l \kappa_a e^{\frac{T_a}{T_{max}}} \left(\frac{\Omega a}{\delta}\right)^{\alpha} (p_1 - S_1)$$

For our results this gives 0.41kN, which is less than a quarter of the previous value. As expected the boundary layer theory does not seem appropriate to use here.

13 Consequences of the full heat equation

The full heat equation is:

$$\rho c_p \left(\frac{\partial T}{\partial t} + (\mathbf{u}.\nabla)\, T\right) = k\nabla^2 T + \sigma_{ij} e_{ij}$$

In our models we have neglected the left hand side of this equation. We assume we have a steady state, and so we always neglect the $\frac{\partial T}{\partial t}$ term. However we can use this term to give an estimate for the parameter R in our analysis. Just considering diffusion of heat we have:

$$\rho c_p \frac{\partial T}{\partial t} = k\nabla^2 T$$

So in a time t an estimate for the distance R heat flows is given by:

$$R = \sqrt{\frac{kt}{\rho c_p}}$$

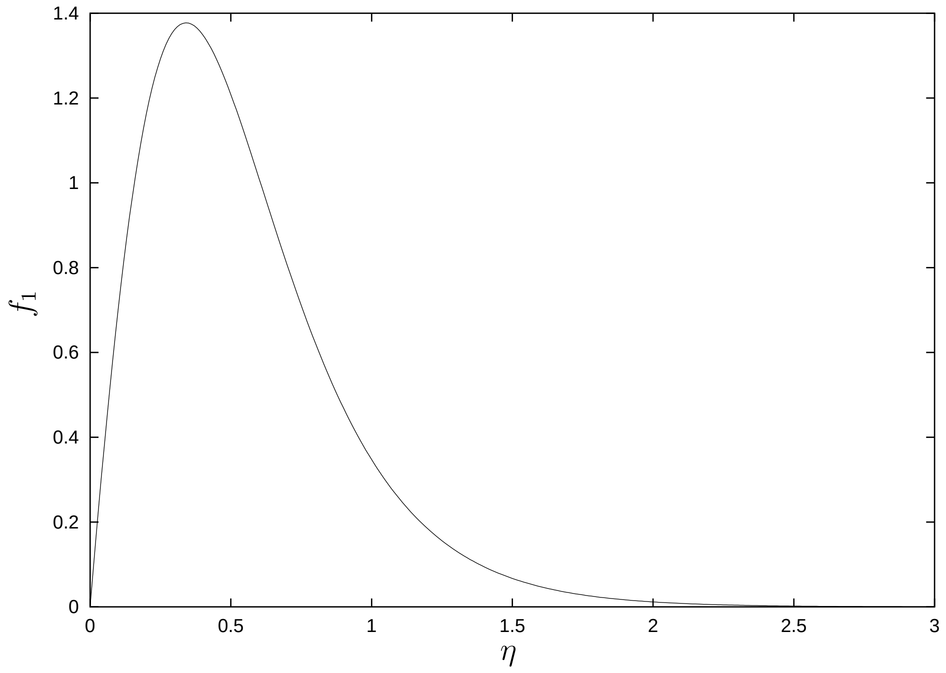

Figure 16: f_1 profile for the exponential boundary layer

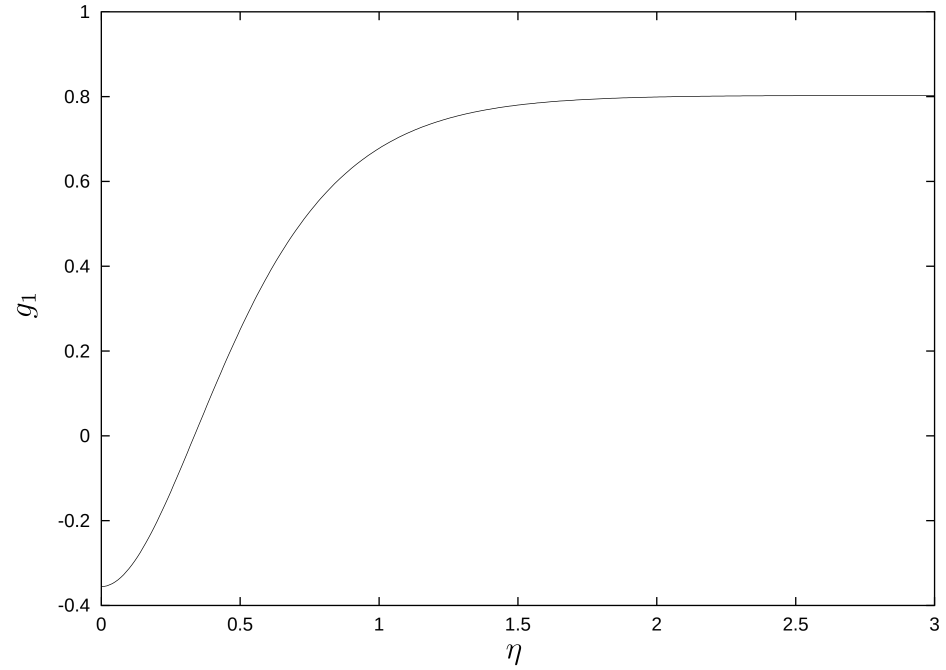

Figure 17: g_1 profile for the exponential boundary layer

Taking as a rough guide a time of 1 minute, this gives figures of $R = 0.076$m for aluminium and $R = 0.023$m for titanium. For convenience $R = 0.1$m has been used throughout, but note that the dependence on R is logarithmic so these differences are not too significant.

For the rotational problem T is only a function of r, and $\mathbf{u}$ is only in the θ direction and so $(\mathbf{u}.\nabla)\,T = 0$. For the translational problem we do need to examine the $(\mathbf{u}.\nabla)\,T$ term carefully. Its behavior is governed by a non-dimensional Péclet number:

$$P = \frac{\rho c_p V a}{k} \tag{9}$$

For small values of the Péclet number the $(\mathbf{u}.\nabla)\,T$ term can be safely neglected. For our data we find values of 0.05 for aluminium and 0.53 for titanium. As such we may be justified in neglecting this term for our aluminium analysis, although for the titanium we may not be justified, and a smaller tool size or weld velocity should be considered.

14 Conclusions

A very idealised model of friction stir welding has been demonstrated that can predict a number of observables. It seems that boundary layer theory is inappropriate to use for generating numerical results but still provides a good general description of the underlying dynamics of the problem. How close these results come to experimental results is unknown, and better numerical data is needed before reasonable predictions can be made.

There are certainly a number of ways this model can be extended. One approach would be to consider 3-D models, for example a simple case would be a rotating hemisphere as a model for the tool. The big advantage in moving to 3-D would be the removal of the log problem in the heat equation. Additionally it would be useful to examine the transient problem to see how close we get to the steady state we have assumed. A rotating transient model would also give a description of the initial phase of the friction stir welding process where the tool drives between the two metals. Further refinements in 3-D would take into account the tool geometries but hopefully this much simplified model can predict the general trends without the need for heavy computation. Further refinements to this model would involve putting back in the $(\mathbf{u}.\nabla)\,T$ term in the heat equation which may produce a significant effect, particularly in the case of harder metals.

References

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[2] H.J. Frost and M.F. Ashby. *Deformation Mechanism Maps: The Plasticity and Creep of Metals and Ceramics*. Pergamon Press, 1982.

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A Numerical example data

All the numerical data used in this document has been based on pure metals, not the alloys commonly used in practice. Particularly note that the κ_a values have been based on a linear extrapolation of the data for the shear modulus at room temperature given in [2] so this can only be used as a rough guide. Furthermore for convenience α for both metals has been set at 0.25, whereas more accurate values would be in the region of 0.23 for titanium and 0.22 for aluminium. Additionally R has been set at 0.1m for both cases for convenience.

The geometry of the weld equipment has been taken to be the same for both metals, namely a tool of radius $a = 0.005$m, angular velocity $\Omega = 30\text{s}^{-1}$, and translational velocity $V = 0.001\text{ms}^{-1}$. The temperature at R is taken to be $T_\infty = 300$K.

The metal data is as follows:

	Aluminium	Titanium
Melting Temperature, T_m (K)	933	1933
Activation Temperature, T_a (K)	2241	6769
α	0.25	0.25
Thermal Conductivity, k ($\text{kgms}^{-3}\text{K}^{-1}$)	235	22
Density, ρ (kgm^{-3})	2700	4507
Specific Heat Capacity, c_p ($\text{m}^2\text{s}^{-2}\text{K}^{-1}$)	900	521
Hardness, κ_a (Pas^α)	1.23×10^6	8.00×10^6